ON NON-STEADY GAS FLOWS ADJACENT TO THE REGIONS OF REST

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It is well known that the one-dimensional plane non-steady flow of a barotropic gas, adjacent to a region of gas at rest, must be a simple wave (cf. [1]). In this case, the region of the perturbed gas and the region of rest are separated by a weak plane discontinuity, along which the first or higher derivatives of the density and velocity components experience discontinuities.

However, in problems where the flow adjoins a region of rest across an arbitrary curved weak discontinuiti in two-dimensional cases, or across some curved weak-discontinuity surface in three-dimensional cases, the flow is, generally speaking, no longer a simple wave. This situation already follows from the fact that in case of single waves, the level surfaces of the main gas-dynamic quantities can either be straight lines (in two-dimensional flows), or plane surfaces (in three-dimensional flows, cf. [2] and [3]).

The attempt to describe the flows behind the arbitrary weak discontinuities by means of the theory of double or triple waves(cf.[4 and 7]) also does not succeed in general, as we shall see below. It is only possible to say in the most general case, that the flow adjacent to a region of rest across a weak discontinuity (which shall in the following be assumed to be sufficiently smooth), is a potential flow. This property may readily be deduced, for example, from the kinematic compatibility conditions, which must be satisfied along the discontinuity surface.

If however, it is assumed in the plane case that the flow satisfies some conditions of smoothness in the neighborhood of the weak discontinuity, then it is possible to use double waves to obtain an approximate solution in some neighborhood of an arbitrary curved weak discontinuity. It is the aim of the present paper to study the equations of the double waves arising in such regions, and also to study the question of applying the results obtained from the use of double waves to describe the flow pattern in the neighborhood of an arbitrary weak discontinuity. At the same time, we shall also consider cases where the derivatives of the density ρ and of the velocity components u_i along the weak discontinuity are not small, and where consequently, the acoustic approximations are inadequate.

We observe that in the three-dimensional case, double waves become automatically unusable, since the level surfaces of ρ and u_i for double wave solutions can only be ruled surfaces; the study of triple waves is difficult since they are described by an over-determined system of nonlinear partial differential equations of complicated structure.

Below we shall study plane isentropic flows of a polytropic gas with the equation of state

$$p = A\rho^{\gamma}, \qquad A = \text{const}$$

These flows are described by Equations

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x_i} = 0 \qquad (i = 1, 2) \tag{0.1}$$

$$\frac{2}{\gamma - 1}\left(\frac{\partial c}{\partial t} + u_1 \frac{\partial c}{\partial x_1} + u_2 \frac{\partial c}{\partial x_2}\right) + c\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = 0 \tag{0.2}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \qquad \left(c^2 = \frac{dp}{d\rho}\right) \tag{0.3}$$

Here p is the pressure, γ the adiabatic exponent, and c the speed of sound.

We shall assume that in the region of rest ($u_i = 0$) and $c = 1$. The u_i in the perturbed region are referred to the unperturbed sound speed, t is in seconds, and the units of x_i are then chosen suitably.

1. To study the behavior of the solutions to the equations of double waves in the neighborhood of the point $u_i = 0$ $(i = 1, 2)$, we shall write the system of equations describing the double wave [4 and 5] in polar coordinates

$$\frac{\gamma - 1}{2}\theta\left[\theta_{rr}\left(1 - \frac{\theta_\varphi^2}{r^2}\right) + \frac{1 - \theta_r^2}{r^2}\theta_{\varphi\varphi} + 2\frac{\theta_r\theta_\varphi}{r^2}\theta_{r\varphi} + \right. \tag{1.1}$$

$$\left. + \frac{\theta_r}{r}(1 - \theta_r^2) - 2\frac{\theta_r\theta_\varphi^2}{r^3}\right] + \frac{\gamma - 3}{2}\left(\theta_r^2 + \frac{\theta_\varphi^2}{r^2}\right) + 2 = 0$$

$$\Phi_{rr}\left(1 - \frac{\theta_\varphi^2}{r^2}\right) + \frac{1 - \theta_r^2}{r^2}\Phi_{\varphi\varphi} + 2\frac{\theta_r\theta_\varphi}{r^2}\Phi_{r\varphi} + \frac{\Phi_r}{r}(1 - \theta_r^2) - 2\frac{\theta_r\theta_\varphi}{r^3}\Phi_\varphi = 0 \tag{1.2}$$

$$x_1 = \left[r\cos\varphi + \frac{\gamma - 1}{2}\theta\left(\theta_r\cos\varphi - \theta_\varphi\frac{\sin\varphi}{r}\right)\right]t + \Phi_r\cos\varphi - \Phi_\varphi\frac{\sin\varphi}{r}$$

$$x_2 = \left[r\sin\varphi + \frac{\gamma - 1}{2}\theta\left(\theta_r\sin\varphi + \theta_\varphi\frac{\cos\varphi}{r}\right)\right]t + \Phi_r\sin\varphi + \Phi_\varphi\frac{\cos\varphi}{r} \tag{1.3}$$

$$\left(u_1 = r\cos\varphi,\ u_2 = r\sin\varphi,\ \theta = \frac{2}{\gamma - 1}c,\ \varphi \in [-\pi, \pi]\right)$$

Here Φ is the velocity potential, and subscripts r and φ denote differentiation with respect to r and φ respectively.

Equations (1.1) and (1.2) are readily obtained from the system (0.1) to (0.3) with the aid of the Cauchy integral (cf. for example, [7]) under the assumption that there exists a relationshi $c = c(u_1, u_2)$, and that u_1 and u_2 are functionally independent (in [4 and 5], potential flow was not assumed). Equations (1.3) are used to construct the flow in the physical space x_1, x_2, t after determining the functions θ and Φ.

The point $u_1 = u_2 = 0$ in the hodograph plane (u_1, u_2 plane) corresponds to the segment of the axis $r=0$, in the coordinates r and φ. It is convenient to change to polar coordinates in the equations of double waves, since if we were to consider them in the u_1, u_2 plane directly, in which plane the equations

$$x_i = \left(u_i + \frac{\gamma-1}{2}\theta\theta_i\right)t + \Phi_i \qquad \left(\theta_i = \frac{\partial\theta}{\partial u_i},\ \Phi_i = \frac{\partial\Phi}{\partial u_i}\right)$$

correspond to equations (1.3), then it would be necessary to seek a solution to the double wave equations, which would be multivalued in θ_i or Φ_i at the point (0,0).

The limiting values of θ_i and Φ_i would then depend on the nature of the approach of (u_1, u_2) to the point (0,0), since otherwise, the point $u_1 = 0$, $u_2 = 0$, $\theta = 2/(\gamma-1)$ would not correspond to some surface of weak discontinuity $\omega(x_1, x_2, t) = 0$ in the physical plane.

We shall seek a solution to the system (1.1) and (1.2) which would satisfy some conditions at $r=0$, in such a way that the formula (1.3) would define in the x_1, x_2, t space some nondegenerate surface $x_1 = x_1(t, \varphi)$,

$$x_2 = x_2(t, \varphi) \text{ and when } r = 0,\ \theta = 2/(\gamma-1)$$

From the kinematic compatibility conditions (cf. [8]) for the function $f(\xi_1, \xi_2, \xi_3)$, whose first derivatives possess discontinuities across the surface $S(\xi_1, \xi_2, \xi_3) = 0$

$$\left[\frac{\partial f}{\partial \xi_k}\right] = h_f \frac{\partial S}{\partial \xi_k}$$

Here h_f is a proportionality factor. Substituting for f the functions u_i and c, and assuming that $\xi_1 = x_1$, $\xi_2 = x_2$, $\xi_3 = t$, we see that the rank of the matrix R for the present case

$$R = \begin{Vmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial t \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial t \\ \partial c/\partial x_1 & \partial c/\partial x_2 & \partial c/\partial t \end{Vmatrix} \tag{1.4}$$

is equal on the discontinuity surface to unity. Thus, at the points on the surface of weak discontinuity $\omega(x_1, x_2, t) = 0$, the quantities c, u_1 and u_2 are pairwise dependent.

We shall further assume that outside the surface of weak discontinuity u_1 and u_2 are functionally independent, i.e. the neighborhood of the point (0,0) in the hodograph plane, corresponding in the class of double waves to the flow region in the x_1x_2t space near the discontinuity surface, can be mapped in one-to-one (except for the point (0,0)) correspondence on the plane $r\varphi$.

Assuming that equations (1.3) for $r=0$ determine some surface, while x_i in (1.3) continuously depend on r and φ (also at $r=0$) we immediately obtain from (1.3) for

$r = 0$ the conditions

$$\lim \frac{\theta_\varphi}{r} = l_1(\varphi), \qquad \lim \frac{\Phi_\varphi}{r} = l_2(\varphi) \quad (l_1(\varphi) \text{ and } l_2(\varphi) \text{ — are continuous}) \tag{1.5}$$

It can be shown that if the derivatives of u_i and c experience finite jumps on the weak discontinuity surface $\omega(x_1, x_2, t) = 0$ and the flow across the weak discontinuity adjoins the state of rest then, at any instant $t = t_0$, the tangents to the instantaneous streamline at the points of the weak discontinuity surface are orthogonal to this surface. This fact does not depend on whether or not the flow is a double wave.

To prove this, we shall write the equation of the streamline at the instant $t = t_0$ as

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} \tag{1.6}$$

Along the stream line $x_1 = x_1(\xi, t_0)$, $x_2 = x_2(\xi, t_0)$ (ξ is the parameter), we have $u_1 = u_1(\xi, t_0)$, $u_2 = u_2(\xi, t_0)$. Let $\xi = \xi_0$ correspond to the point on the surface of the weak discontinuity, and

$$\lim_{\xi\to\xi_0} \frac{u_1}{u_2} = \lim_{\xi\to\xi_0} \frac{u_1'}{u_2'} = \lim_{\xi\to\xi_0} \frac{dx_1}{dx_2} = \left(\frac{dx_1}{dx_2}\right)_{\xi=\xi_0} = a \tag{1.7}$$

where the prime denotes differentiation with respect to ξ. From (1.7), with help of the kinematic conditions of compatibility, we obtain for the derivatives u_1 and u_2 the following

$$\frac{u_1'}{u_2'} = \frac{x_1'\,\partial u_1/\partial x_1 + x_2'\,\partial u_1/\partial x_2}{x_1'\,\partial u_2/\partial x_1 + x_2'\,\partial u_2/\partial x_2} \to \frac{ah_1\,\partial\omega/\partial x_1 + h_1\partial\omega/\partial x_2}{ah_2\,\partial\omega/\partial x_1 + h_2\partial\omega/\partial x_2} \qquad (\xi \to \xi_0) \tag{1.8}$$

From this, $a = h_1 / h_2$. But, from the condition of potential flow (0.3), it follows that

$$\frac{h_1}{h_2} = \frac{\partial\omega/\partial x_1}{\partial\omega/\partial x_2}$$

on the weak discontinuity surface, so that the assertion is proved for the case when the instantaneous streamline approaches the discontinuity surface and is not tangent to it. The case of tangency, when $a(\partial\omega/\partial x_1) + \partial\omega/\partial x_2 = 0$, cannot be realized, since if along the trajectory $u_1 \not\equiv 0$ and $u_2 \not\equiv 0$, then we always have

$$a = \lim \frac{u_1'}{u_2'} = \frac{h_1}{h_2}$$

For $a = 0$ and $a = \infty$ the consideration is carried out in a similar manner.

For the normal velocity of propagation V of the weak discontinuity, whose motion is governed by (1.3), we have

$$V = \left|\frac{\partial x_1}{\partial t}\cos\varphi + \frac{\partial x_2}{\partial t}\sin\varphi\right| = \left|\frac{\gamma - 1}{2}\,\theta\theta_r\right| = 1 \tag{1.9}$$

since the weak discontinuity propagates with the speed of sound, equal in this case to unity, and the vector $(\cos\varphi, \sin\varphi)$ is normal to the discontinuity surface. From (1.9), we readily obtain for $r = 0$,

$$|\theta_r| = 1 \tag{1.10}$$

Thus, we have for equation (1.1) on the line $r = 0$ the initial Cauchy relations

$$\theta = \frac{2}{\gamma - 1}, \quad \theta_\varphi = 0, \quad |\theta_r| = 1 \quad \text{for} \quad r = 0 \tag{1.11}$$

The quantity θ_r is continuous at $r = 0$, hence two cases must always be considered: $\theta_r = 1$ and $\theta_r = -1$.

Let us now consider the class of solutions of the equations of double waves, for which all the fourth order derivatives of the function θ, together with the mixed derivatives (second order in both r- and φ) are continuous in the neighborhood of $r = 0$.

We observe that this class includes all the self-similar cylindrical flows with the similarity variable $\sqrt{x_1^2 + x_2^2} / t$. The behavior of these flows, in the neighborhood of a weak discontinuity which leaves behind it a region of rest, was studied, for example, in [9]. In contrast to the one-dimensional spherical motion, when on the weak discontinuity (cf [9]) (motion of weak discontinuity behind normal detonation wave – at the front of the wave the Chapman-Jouguet condition is satisfied) first derivatives of u_i and ρ remain in such situations continuous while the second and all the other derivatives become infinite, in the cylindrical case (similarly to the one-dimensional plane case), the first derivative already has a first order discontinuity*.

Let us consider the neighborhood of $r = 0$, $\Delta r = h$, $h < 1$. In accordance with the assumptions made, we have in this neighborhood

$$\theta(r, \varphi) = \theta(0, \varphi) + \theta_r(0, \varphi) r + \theta_{rr}(r_0, \varphi) r^2 / 2 \quad (0 < r_0 < r) \tag{1.12}$$

Using the continuity of $\theta_{rr\varphi\varphi}$ at $r = 0$, we obtain the estimates

$$\frac{\theta_\varphi}{r} \sim O(h), \quad \frac{|1 - \theta_r^2}{r} \sim O(1), \quad \frac{\theta_{\varphi\varphi}}{r} \sim O(h), \quad \frac{\theta_{r\varphi}}{r} \sim O(1) \tag{1.13}$$

Neglecting in (1.1) terms of the order of $O(h)$ and $O(h^2)$, and keeping the terms of the order $O(1)$, we obtain for θ the equation

$$\frac{\gamma - 1}{2} \theta \left[\theta_{rr} + \frac{\theta_r (1 - \theta_r^2)}{r}\right] + \frac{\gamma - 3}{2} \theta_r^2 + 2 = 0 \tag{1.14}$$

Conditions (1.11) determine, in the neighborhood of $r = 0$, a family of solutions of Equation (1.14), consequently Equation (1.1) also possesses a family of solutions

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* In [10] a method is shown for constructing a series of exact solutions behind two-dimensional normal detonation waves. These solutions were such, that at some distance behind the wave, there was a moving curved weak discontinuity, leaving behind it a region of rest. Because of some arithmetical error in determining the behavior of the expression $\theta_r^2 - 1$ near $r = 0$ (we have put $\theta_r^2 - 1 \sim Br$, while it should have been $\theta_r^2 - 1 \sim B\sqrt{r}$, $B = \text{const}$) we incorrectly stated there that the first derivatives of u_i and c are continuous on the weak discontinuity. However, all the further basic results, and the stated boundary value problems, remain correct. All the calculations are also carried out in a completely similar manner.

satisfying (1.11), since Equation (1.14) can be obtained from (1.1) simply by assuming that θ does not depend on φ.

In fact, let us first consider the case $\theta_r = 1$. Assuming that $y = \theta_r$ and carrying out a linearization in (1.14) in the neighborhood of $r = 0$, we obtain for $y(r)$ the equation

$$y_r + \frac{2(1-y)}{r} + \frac{\gamma+1}{2} = 0 \tag{1.15}$$

From this $y = 1 + {}^1/_2(\gamma + 1)r + C(\varphi)r^2$, and

$$\theta = \frac{2}{\gamma - 1} + r + \frac{\gamma+1}{4}r^2 + \frac{1}{3}C(\varphi)r^3 \tag{1.16}$$

Here $C(\varphi)$ is an arbitrary function of φ.

Similarly, for $\theta_r = -1$ we have

$$\theta = \frac{2}{\gamma - 1} - r + \frac{\gamma+1}{4}r^2 + \frac{1}{3}C(\varphi)r^3 \tag{1.17}$$

From (1.16) and (1.17), it follows that the following expression is uniquely determined in the neighborhood of $r = 0$, with the accuracy of $O(h)$

$$\frac{1 - \theta_r^2}{r} = 2A \qquad \left(A = \pm\frac{\gamma+1}{2} \quad \text{for} \quad \theta_r = \mp 1\right) \tag{1.18}$$

The signs are assumed to be in correspondence. In addition, we always have $\theta_{rr} = {}^1/_2(\gamma + 1)$ for $r = 0$. Assuming that in (1.2), Φ is twice continuously differentiable with respect to r and φ, using the estimates for small r

$$r^{-1}\theta_\varphi \sim O(h^2), \qquad \Phi_\varphi \sim O(h), \qquad \Phi_{\varphi\varphi} \sim O(h)$$

and the relation (1.18), and neglecting in (1.2) the terms of the orders $O(h)$ and $O(h^2)$, we finally obtain for Φ

$$r\Phi_{rr} + 2A\Phi_{\varphi\varphi} + 2Ar\Phi_r = 0 \tag{1.19}$$

Depending on the sign of A, equation (1.19) in the half-plane $r > 0$ may either be elliptic or hyperbolic. If the density of the gas increases with the distance from the weak discontinuity surface, $\theta_r = 1$ (e.g. weak discontinuities behind a normal detonation wave), $A < 0$, and equation (1.19) is hyperbolic for $r > 0$. Conversely, when the density decreases (when a weak discontinuity moves across the gas at rest, picking up new masses of gas), $\theta_r = -1$, $A > 0$ and equation (1.19) will be elliptic in type. In both cases, the line $r = 0$, on which $\Phi_\varphi = 0$ ($\Phi = \text{const}$), will be a degenerate parabolic line; at the same time, it will be a characteristic line, since for $dr = 0$, $d\Phi_\varphi = 0$ the conditions of a characteristic strip (cf. [8]) are satisfied for Equation (1.19). We note that the Cauchy relations (1.11) also define a characteristic strip, while the line $r = 0$ is a line along which the equations (1.1) and (1.2) are parabolic.

The substitution $r = \mp z / 2A$, reduces Equation (1.19) in the hyperbolic case to the form

$$z\Phi_{zz} - \Phi_{\varphi\varphi} - z\Phi_z = 0 \qquad (z > 0) \tag{1.20}$$

and in the elliptic case, to the form

$$z\Phi_{zz} + \Phi_{\varphi\varphi} + z\Phi_z = 0 \qquad (z > 0) \tag{1.21}$$

Let us consider the question of determining the initial data for $r = 0$ for the function Φ, so as to have the formulas (1.3) define the motion of a weak discontinuity of arbitrary form. We shall note that the shape of the weak discontinuity surface in the x_1, x_2, t space is determined as soon as we are given the shape of the line of intersection of this surface with the plane $t = 0$. Without loss of generality, we shall assume that the motion of a weak discontinuity is given by the equation

$$t = \Psi\ (x_1,\ x_2) \tag{1.22}$$

The function Ψ satisfies the equation

$$\Psi_1^2 + \Psi_2^2 = 1, \qquad \Psi_i = \partial\Psi / \partial x_i \tag{1.23}$$

since the front of the weak discontuity moves with a constant normal velocity, equal to the speed of sound. Equation (1.23) can be easily integrated for arbitrary initial data, and consequently, the motion of the weak discontinuity will be determined.

Let us put $\Phi_r = \Pi\ (\varphi)$ at $r = 0$. Then, by the assumptions made above, $\lim\ (r^{-1}\Phi_\varphi) = l_2\ (\varphi) = \Pi'\ (\varphi)$ as $r \to 0$ (cf. (1.5)), and from (1.3) it follows that the front of the weak discontinuity is, at $t = 0$, given by the parametric equations

$$x_1 = \Pi\ (\varphi) \cos\varphi - \Pi'\ (\varphi) \sin\varphi, \qquad x_2 = \Pi\ (\varphi) \sin\varphi + \Pi'\ (\varphi) \cos\varphi \tag{1.24}$$

If the line of weak discontinuity is given at $t = 0$ by the equation $F\ (x_1,\ x_2) = 0$, then we have for $\Pi\ (\varphi)$ the ordinary differential equation

$$F\ (\Pi\ (\varphi) \cos\varphi - \Pi'\ (\varphi) \sin\varphi,\ \ \Pi\ (\varphi) \sin\varphi + \Pi'\ (\varphi) \cos\varphi) = 0 \tag{1.25}$$

Using the geometric relations at $\varphi = 0$ to obtain some point $(x_1,\ x_2) = (a,\ b)$, we can write the initial condition for (1.25) in the form $\Pi\ (0) = a$.

Thus, we have for Equation (1.19) the initial relations

$$\Phi = 0, \quad \Phi_\varphi = 0, \quad \Phi_r = \Pi\ (\varphi) \quad \text{for} \quad r = 0 \tag{1.26}$$

since the additive constant in Φ is unimportant.

We note that, finding from (1.3) for $r = 0$, $t = t_0$ the vector tangent to the weak discontinuity $(\partial x_1 / \partial\varphi,\ \partial x_2 / \partial\varphi)$ we have with the aid of (1.24),

$$\begin{aligned} \partial x_1 / \partial\varphi &\to \sin\varphi\ (-\ t_0\theta_r - \Pi\ (\varphi) - \Pi''\ (\varphi)) \\ \partial x_2 / \partial\varphi &\to \cos\varphi\ (t_0\theta_r + \Pi\ (\varphi) + \Pi''\ (\varphi)) \end{aligned} \qquad \text{for} \quad r \to 0 \tag{1.27}$$

i.e. the vector $(\cos\varphi,\ \sin\varphi)$, in accordance with the above, is a unit vector normal to the discontinuity line.

2. Let us study the Cauchy problem at $z = 0$ for the equations (1.20) and (1.21). Generally speaking, a problem with initial data corresponding to the parabolic case is incorrect for both elliptic and hyperbolic cases [11].

Let us first consider the hyperbolic case, and using the theorem of [12], we shall show that there exists a twice continuously differentiable solution to the problem posed for Equation (1.20), and that such a solution is unique.

Introducing the functions

$$\Phi_z = u, \qquad \Phi_\varphi = -v \tag{2.1}$$

we shall transform Equation (1.20) to a system of two equations of first order

$$u_\varphi + v_z = 0, \qquad zu_z + v_\varphi - zu = 0 \tag{2.2}$$

Initial data for u and v will be

$$u = \frac{1}{\gamma+1}\Pi(\varphi), \quad v = 0 \quad \text{for} \quad z = 0 \tag{2.3}$$

Let the region D be bounded by the segment $[M, N]$ of the φ-axis (points M and N belong to the interval $[-\pi, \pi]$) and the characteristics

$$\varphi - 2\sqrt{z} = C_1 = \text{const}, \qquad \varphi + 2\sqrt{z} = C_2 = \text{const} \tag{2.4}$$

passing through the points M and N respectively. The coefficients of the system (2.2) are continuous in the rectangle $\varphi \in [M, N]$, $z \in [0, \delta]$, where δ is such, that this rectangle contains the region D, including its boundaries.

Calculating the function (the initial conditions of Theorem 1 [12] have the form $v(\varphi, 0) = \nu(\varphi)$, and $\eta^{-1}(u - P) = \tau(\varphi)$ on MN, where P is some fixed function (see [12]), and ν and τ are sufficiently smooth):

$$\eta(\varphi, z) = \exp\int_z^\delta b(\varphi, \tau)\,\tau^{-1}\,d\tau$$

where b is the coefficient of u in the second equation of (2.2), we shall have $\eta = \exp(z - \delta)$, i.e. $\eta = \text{const} \neq 0$ at $z = 0$. By Theorem 1 of [12], whose conditions are satisfied in the present case, and assuming that $\Pi(\varphi)$ has four continuous derivatives, we obtain the existence of an unique twice continuously differentiable solution of system (2.2) with initial data (2.3).

Putting $\Phi = \chi(z)\psi(\varphi)$, in (1.20), we obtain the ordinary differential equations for χ and ψ

$$\psi'' + \lambda\psi = 0 \tag{2.5}$$

$$z\chi'' - z\chi' + \lambda\chi = 0 \tag{2.6}$$

where λ is an undetermined parameter. Just as in [10], it is easy to show that two linearly independent solutions of equation (2.6) in the neighborohood of $z = 0$ must have the form

$$\chi_1 = A_1 z + o(z), \qquad A_1 = \text{const} \neq 0$$
$$\chi_2 = A_2 + o(1), \qquad A_2 = \text{const} \neq 0$$

From (1.26), it follows that $X(0) = 0$, and X_2 cannot be a solution.

For χ_λ at $z = 0$, the following initial data may be taken

$$\chi_\lambda(0) = 0, \qquad \chi_\lambda'(0) = 1 \tag{2.7}$$

We shall consider a closed weak discontinuity, with $\varphi \in [-\pi, \pi]$; from (2.5), we obtain $\lambda = k^2$ $(k = 0, 1, 2 \ldots)$. Putting $\chi = zy$, we have from (2.6) the following equation for $y(z)$

$$zy'' + (2 - z)y' + (k^2 - 1)y = 0 \tag{2.8}$$

Assuming that y'' is continuous at $z = 0$, from (2.8), we have, for the values of z near to zero,

$$y = \exp\left[-{}^1/_2 (k^2 - 1)z\right] \tag{2.9}$$

hence, the expression for Φ in some small neighborhood of $z = 0$ is the following

$$\Phi = z \sum_k (a_k \cos k\varphi + b_k \sin k\varphi) \exp\left(-\frac{k^2 - 1}{2} z\right) \tag{2.10}$$

where

$$\sum_k a_k \cos k\varphi + b_k \sin k\varphi = \frac{1}{\gamma + 1} \Pi(\varphi) \tag{2.11}$$

It is clear that for sufficiently smooth functions $\Pi(\varphi)$, the application of Fourier's method is correct.

Let us consider the elliptic case. The problem with Cauchy data (2.3) for equation (1.21) is incorrect in the classical sense.

Equation (1.21) belongs to the class of equations, whose boundary value problems have been studied by Kel'dysh [13], who considered the boundary value problems in a region bounded by the segment MN of the φ-axis and a smooth curve Γ, based on the segment MN and located in the half-plane $z > 0$. Existence of a continuous solution of the Dirichlet problem and of the problem E (when segment MN is free of given boundary conditions) depends on the behavior of the coefficient $b(\varphi, z)$ of Φ_z at $z = 0$. Since in the present case $b(\varphi, 0) = 0$, then by the theorem of Kel'dysh, there exists some continuous solution to the Dirichlet problem, and the contour Γ may be chosen arbitrarily (except that it must be smooth), and also Φ may be arbitrarily given on Γ ($\Phi = 0$ on MN). Each such solution generates some function $\Pi(\varphi)$ and consequently, some surface of weak discontinuity.

In this manner, we can obtain a family of solutions, depending on the arbitrary functions shown above, and then we can try to use them to approximate the given function $\Pi(\varphi)$. We can also immediately solve the Cauchy problem (2.3) for equation (1.21) by Fourier methods exactly as it was done in the hyperbolic case, taking relatively few terms of the series, i.e. using the simplest method of regularisation of an incorrect problem. The existence of an exact solution of the given problem remains an open question.

Till now, we considered the behavior of double waves only in the neighborhood of the line $r = 0$ and introduced the equations describing the motion of gas in the narrow vicinity of the weak discontinuity.

However, equation (1.14), as observed before, is exact in the sense that it can be obtained from the equations of double waves under the single assumption that $\theta = \theta(r)$. This equation can also be obtained from the equations of cylindrical self-similar motion with the independent variable $\sqrt{x_1^2 + x_2^2} / t$ (cf [14]).

For such a function θ, equation (1.2) will have the form

$$r^2\Phi_{rr} + (1 - \theta_r^2)\Phi_{\varphi\varphi} + r(1 - \theta_r^2)\Phi_r = 0 \tag{2.12}$$

(this equation was studied in [10], but for other purposes).

With the aid of the systems of equations (1.14) and (2.12), whose solutions satisfy the initial conditions stated above, we can find the family of solutions of the gas dynamical equations (for example, by Fourier's method), belonging to the class of double waves, behind the surface of a weak discontinuity of, in general, arbitrary shape. These solutions already are no longer valid only in the vicinity of the weak discontinuity, but are correct in the general case until the appearance of limiting lines in the flow (cf [10]). In the neighborhood of the discontinuity these flows behave in the same manner as the flows obtained from the systems (1.20) and (1.21).

In section 3 we show that under some conditions, the flow behind the surface of an arbitrary weak discontinuity can, from some definite instant of time, be approximately treated as a double wave. Thus, the formalism of double waves gives an effective means to obtain approximate flow picture in some neighborhood of an arbitrary weak discontinuity of the type described.

3. We shall assume that a weak discontinuity surface, propagating into a gas at rest, is given by equation (1.22), where the function Ψ satisfies (1.23). Surface (1.22), will be a characteristic surface of the systems of equations (0.1) and (0.2), on which $u_i = 0$ and $c = 1$.

The jumps arising in the derivatives $u_{i\psi}$ and c_ψ across the characteristic surface (1.22) satisfy (cf [15]) the relation

$$(u_{1\psi},\ u_{2\psi},\ c_\psi) = \sigma\,(c\Psi_1,\ c\Psi_2,\ {}^1/_2\,(\gamma - 1)) \tag{3.1}$$

where $u_{i\psi}$ and c_ψ are the jumps in the derivatives normal to the surface (1.22), the vector $(c\Psi_1,\ c\Psi_2,\ {}^1/_2\,(\gamma - 1))$ is the right null-vector of the characteristic matrix of the systems (0.1) and (0.2), and σ is some scalar multiplier. The characteristic surface (1.22) may be covered by a family of bi-characteristic rays, satisfying the equations

$$t^{\cdot} = 1,\ \ x_1^{\cdot} = c^2\Psi_1, \qquad x_2^{\cdot} = c^2\Psi_2 \tag{3.2}$$

In the present case, we take time t as the parameter on the bi-characteristic ray, along which the derivative is taken (denoted by a dot in (3.2)). The weak discontinuity surface (1.22), determined by Equations (1.23), will be a developable surface in $x_1 x_2 t$, in this case, while the bi-characteristics, which are characteristics of equation (1.23), are in this case, straight lines in x_1x_2t. Along the bi-characteristics, $\Psi_1 = \text{const}$ and $\Psi_2 = \text{const}$.

This follows from the relation

$$\Psi_1 x_1^{\cdot} + \Psi_2 x_2^{\cdot} = 1$$

which shows that the surface $t = \Psi(x_1, x_2)$ is an integral of the system (3.2).

In [15], for systems of first order linear equations, an ordinary differential equation (equation of transport) was obtained, in accordance with which the scalar σ propagates along the bi-characteristic rays, and the possibility was indicated of obtaining the equations of transport for quasilinear systems. The equation of transport for two quasilinear equations with two independent variables (with σ propagating along the characteristics) was studied in detail in [16]. Below, we shall derive the equation of transport for the system (0.1) and (0.2) in the case of a flow adjacent to a region of rest. This will be utilised later.

Let us change to new independent variables in (0.1) and (0.2).

$$\xi_1 = x_1, \qquad \xi_2 = x_2, \qquad \xi_3 = \Psi(x_1, x_2) - t \tag{3.3}$$

On substitution, we obtain

$$(-1 + u_1\Psi_1 + u_2\Psi_2)\frac{\partial u_i}{\partial \xi_3} + u_1\frac{\partial u_i}{\partial \xi_1} + u_1\frac{\partial u_i}{\partial \xi_2} + \frac{2}{\gamma - 1}c\left(\frac{\partial c}{\partial \xi_i} + \frac{\partial c}{\partial \xi_3}\Psi_i\right) = 0 \quad (i = 1, 2) \tag{3.4}$$

$$\begin{gathered}\frac{2}{\gamma-1}(-1 + u_1\Psi_1 + u_2\Psi_2)\frac{\partial c}{\partial \xi_3} + \frac{2}{\gamma-1}\left(u_1\frac{\partial c}{\partial \xi_1} + u_2\frac{\partial c}{\partial \xi_2}\right) + \\ + c\left(\frac{\partial u_1}{\partial \xi_1} + \frac{\partial u_2}{\partial \xi_2} + \frac{\partial u_1}{\partial \xi_1}\Psi_1 + \frac{\partial u_2}{\partial \xi_3}\Psi_2\right) = 0\end{gathered} \tag{3.5}$$

For the system (3.4) and (3.5), the equation of the characteristic surface $R(\xi_1, \xi_2, \xi_3) = 0$ has the form

$$\begin{gathered}\frac{2}{\gamma-1}[R_3(-1 + u_1\Psi_1 + u_2\Psi_2) + u_1R_1 + u_2R_2]\,\{[R_3(-1 + u_1\Psi_1 + u_2\Psi_2) + \\ + u_1R_1 + u_2R_2]^2 - c^2(R_3\Psi_1 + R_1)^2 - c^2(R_3\Psi_2 + R_2)^2\} = 0 \quad \left(R_i = \frac{\partial R}{\partial \xi_i}\right)\end{gathered} \tag{3.6}$$

and the surface $R = \xi_3 = 0$ corresponds to the weak discontinuity surface.

The equations of the bicharacteristics are

$$\xi_1^{\cdot} = c^2\Psi_1, \qquad \xi_2^{\cdot} = c^2\Psi_2, \qquad \xi_3^{\cdot} = 0 \tag{3.7}$$

Let L_1, L_2, and L_3 denote the left-hand sides of the equations obtained by differentiating with respect to ξ_3 (3.4) and (3.5) respectively. Multiplying L_1, L_2, and L_3 by $c\Psi_1$, $c\Psi_2$, and 1, and adding, we have, on the characteristic surface,

$$\begin{gathered}c\Psi_1L_1 + c\Psi_2L_2 + L_3 = \frac{2}{\gamma-1}c^2\left(\frac{\partial^2 c}{\partial \xi_1\partial \xi_3}\Psi_1 + \frac{\partial^2 c}{\partial \xi_2\partial \xi_3}\Psi_2\right) + \\ + c\left(\frac{\partial^2 u_1}{\partial \xi_1\partial \xi_3} + \frac{\partial^2 u_2}{\partial \xi_2\partial \xi_3}\right) + c\Psi_1\left(\Psi_1\frac{\partial u_1}{\partial \xi_3} + \Psi_2\frac{\partial u_2}{\partial \xi_3}\right)\frac{\partial u_1}{\partial \xi_3} + \\ + \frac{2}{\gamma-1}c\Psi_1\frac{\partial c}{\partial \xi_3}\left(\frac{\partial c}{\partial \xi_1} + \frac{\partial c}{\partial \xi_3}\Psi_1\right) + c\Psi_2\left(\Psi_1\frac{\partial u_1}{\partial \xi_3} + \Psi_2\frac{\partial u_2}{\partial \xi_3}\right)\frac{\partial u_2}{\partial \xi_3} +\end{gathered} \tag{3.8}$$

$$+\frac{2}{\gamma-1}c\Psi_2\frac{\partial c}{\partial\xi_3}\left(\frac{\partial c}{\partial\xi_2}+\frac{\partial c}{\partial\xi_3}\Psi_2\right)+\frac{2}{\gamma-1}\left(\Psi_1\frac{\partial u_1}{\partial\xi_3}+\Psi_2\frac{\partial u_2}{\partial\xi_3}\right)+$$
$$+\frac{\partial c}{\partial\xi_3}\left(\frac{\partial u_1}{\partial\xi_1}+\frac{\partial u_2}{\partial\xi_2}+\frac{\partial u_1}{\partial\xi_3}\Psi_1+\frac{\partial u_2}{\partial\xi_3}\Psi_2\right)=0$$

Differentiation with to ξ_1 and ξ_2 will be internal on the characteristic surface $\xi_3 = 0$. Since the weak discontinuity front, all derivatives of the functions u_i and c are equal to zero, then from (3.1), we have for the jumps in the derivatives in this case

$$\left(\frac{\partial u_1}{\partial\xi_3},\ \frac{\partial u_2}{\partial\xi_3},\ \frac{\partial c}{\partial\xi_3}\right)=\sigma\left(c\Psi_1,\ c\Psi_2,\frac{\gamma-1}{2}\right) \tag{3.9}$$

Using (1.23), (3.9) and the relation

$$f^{\cdot}=c^2\Psi_1\frac{\partial f}{\partial\xi_1}+c^2\Psi_2\frac{\partial f}{\partial\xi_2}$$

to differentiate the function f along the bicharacteristics (3.7), we finally obtain from (3.8) the following transport equation for σ :

$$\sigma^{\cdot}+\frac{\gamma+1}{2c}\sigma^2+\frac{c^2(\Psi_{11}+\Psi_{22})}{2}\sigma=0\quad\left(\Psi_{ik}=\frac{\partial^2\Psi}{\partial\xi_i\,\partial\xi_k}\right) \tag{3.10}$$

In the present case, (3.10) is a Riccati equation, in contrast to the case of a linear system, when the transport equation is also a linear ordinary equation. Instead of ξ_1 and ξ_2 in the coefficients of (3.10), we should substitute the quantities $\xi_1 = a_1t + b_1$, $\xi_2 = a_2t + b_2$, $a_i = \text{const}$, $b_i = \text{const}$, $c = 1$. This follows from (3.7) after integrating the bicharacteristic equations.

If the weak discontinuity is planar, i.e. $\Psi_{11} + \Psi_{22} = 0$, then from (3.10), we have the equation

$$\sigma^{\cdot} + {}^1/_2\,(\gamma + 1)\sigma^2 = 0 \tag{3.11}$$

whose general solution is

$$\sigma=\frac{1}{{}^1/_2(\gamma+1)\,t+A}\qquad(A=\text{const}) \tag{3.12}$$

This result corresponds to the theory of simple waves, the class of which always includes a plane one-dimensional flow adjoining a region of rest across a weak discontinuity. In fact, writing the equation of simple waves as (cf. [9])

$$x - [{}^1/_2\,(\gamma + 1)u \pm 1]t = F\,(u)$$

where u is the velocity, x is the space coordinate and $F\,(u)$ is an arbitrary function ($F\,(0) = 0$), and expanding (u) near $u = 0$ into a Taylor series, we shall have for $\partial u / \partial x$, on the weak discontinuity,

$$\frac{\partial u}{\partial x}=\frac{1}{{}^1/_2(\gamma+1)\,t+F_0},\quad F_0=\frac{\partial F}{\partial u}\quad\text{or}\quad u=0$$

Let us consider the one-dimensional cylindrical case, when the motion of the weak discontinuity is defined by the equation $\sqrt{x_1^2 + x_2^2} = t$. In this case, $\Psi_{11} + \Psi_{22} = 1 / t$ and equation (3.10) assumes the form

$$\sigma^{\cdot}+\frac{\gamma+1}{2}\sigma^2+\frac{1}{2t}\sigma=0 \tag{3.13}$$

Using the fact that $\sigma = 1 / (\gamma + 1)t$ is a particular solution, we can write the general solution of (3.13) as

$$\sigma = \frac{1}{(\gamma+1)\,t} + \frac{1}{At^{3/2} - (\gamma+1)\,t} \qquad (A = \text{const}) \tag{3.14}$$

We shall consider some neighborhood Δ_k of the weak discontinuity (1.22), characterized by the fact that at any $t \geqslant t_k$ and for any point (x_1, x_2, t), in this neighborhood, there exists a point $(x_1^\circ, x_2^\circ, t)$ on the surface (1.22), such that

$$|x_1 - x_1^\circ| \leqslant k, \qquad |x_2 - x_2^\circ| \leqslant k$$

From (3.14), it follows that if at some instant $t = t_0$ the weak discontinuity occupies the position $\sqrt{x_1^2 + x_2^2} = t_0$, and two different flows behind it are characterised by two different scalars σ_1 and σ_2 to which correspond the constants $A_1 < \infty$ and $A_2 = \infty$ (i.e. self-similar motion with independent variable $\sqrt{x_1^2 + x_2^2} / t$), then for large t

$$|\sigma_1 - \sigma_2| \sim O\,(t^{-3/2}) \tag{3.15}$$

Thus, if a weak discontinuity propagating into the gas at rest exists in a one-dimensional cylindrical flow, and if at the instant $t = t_0$, it is known that the derivatives of all the gas-dynamical quantities undergo a finite jump and the radius of the weak discontinuity increases with t, then, in the neighborhood Δ_k the given flow may during the time interval $t_k \sim O\,(k^{-2/3})$ be approximated by a self-similar flow with variable $\sqrt{x_1^2 + x_2^2} / t$ with the accuracy $O\,(k^2)$ for $t > t_k$ $(k < 1)$. The requirement of the accuracy $O\,(k^2)$ arises in connection with the determination of u_1, u_2 and c.

It can be shown that a similar result also holds for weak discontinuities of arbitrary form, provided its radius of curvature increases with time. The only difference is that instead of a self-similar solution, the class of double wave solutions must be considered. Also, it is obviously always assumed, that the flow in the neighborhood of the weak discontinuity is sufficiently smooth.

In fact, differentiating (1.23) with respect to u_1 and u_2, we have

$$\Psi_{11} + \Psi_{22} = -\frac{\Psi_{21}}{\Psi_1\Psi_2} \tag{3.16}$$

Introducing the curvature K of the plane curve $\Psi\ (x_1, x_2) = t_0$, lying in the plane $t = t_0$, and using (1.23), we write (3.10) in the form

$$\sigma^{\cdot} + {}^1\!/_2\ (\gamma+1)\,\sigma^2 + {}^1\!/_2\,K\sigma = 0 \tag{3.17}$$

Along the fixed bicharacteristic with $\Psi_1 = \text{const}$, and $\Psi_2 = \text{const}$, the quantity K may be represented in this case as

$$K = \frac{1}{t+B} \qquad (B = \text{const}) \tag{3.18}$$

Indeed, obtaining along the bicharacteristics the derivative

$$\left(\frac{1}{\Psi_{21}}\right)^{\cdot} = -\frac{\Psi_{211}\Psi_1 + \Psi_{212}\Psi_2}{\Psi_{21}^2}, \qquad \Psi_{ikj} = \frac{\partial^3\Psi}{\partial x_i\,\partial x_j\,\partial x_k} \tag{3.19}$$

with the aid of the equality

$$\Psi_{112}\Psi_1 + \Psi_{11}\Psi_{22} + \Psi_{122}\Psi_2 + \Psi_{12}\Psi_{22} = 0$$

which was obtained by differentiating (1.23) with respect to x_1 and x_2, we get

$$\left(\frac{1}{\Psi_{21}}\right)^{\cdot} = -\frac{1}{\Psi_1\Psi_2} \tag{3.20}$$

Integrating (3.20), we obtain for Ψ_{21}

$$\Psi_{21} = \frac{1}{-t\,/\,\Psi_1\Psi_2 + B} \qquad (B = \text{const}) \tag{2.21}$$

Finally, we write (3.10) in the general case as

$$\sigma^{\cdot} + \frac{\gamma+1}{2}\sigma^2 + \frac{1}{2(t+B)}\sigma = 0 \tag{3.22}$$

Its general solution is

$$\sigma = \frac{1}{(\gamma+1)(t+B)} + \frac{1}{A(t+B)^{3/2} - (\gamma+1)(t+B)} \qquad (A = \text{const}) \tag{3.23}$$

From relations (1.3), which in accordance with the above discussion, have the following form in the neighborhood of the weak discontinuity

$$\begin{aligned} x_1 &= [r + {}^1/_2(\gamma-1)\,\theta\theta_r]\cos\varphi t + \Phi_r\cos\varphi - r^{-1}\Phi_\varphi\sin\varphi \\ x_2 &= [r + {}^1/_2(\gamma-1)\,\theta\theta_r]\sin\varphi t + \Phi_r\sin\varphi + r^{-1}\Phi_\varphi\cos\varphi \end{aligned} \tag{3.24}$$

for the scalar σ_d, corresponding to the class of double waves, we use the conditions (1.11) and (1.26) on the weak discontinuity for the functions θ and Φ to obtain the following propagation equation along the bicharacteristics:

$$\sigma_d = \frac{1}{(\gamma+1)(t+B_1)} \qquad (B_1 = \text{const}) \tag{3.25}$$

let us assume that the radius of curvature of a weak discontinuity increases with time, that the perturbations in the flow behind the discontinuity do not catch up with the discontinuity (flow in the neighborhood of the discontinuity is sufficiently smooth) and, that at the time $t = t_0 > |B_1|$, the scalar σ in (3.23) along some arc of the weak discontinuity is determined by the constants A, B, $a_0 \leqslant A \leqslant a_1$, $b_0 \leqslant B \leqslant b_1$, a_0, a_1, b_0, and b_1 = const, in such a way that it does not become infinite with increasing t. Then, on the part of the surface of weak discontinuity $(t > t_0)$, formed by the bicharacteristic passing through the points of the above mentioned arc, we have, for sufficiently large t,

$$|\sigma_d - \sigma| \sim O(t^{-3/2}) \tag{3.26}$$

Thus, in case of an arbitrary weak discontinuity under the above restrictions in the neighborhood Δ_k and for sufficiently large $t_k \sim O(k^{-2/3})$ with accuracy of $O(k^2)$, the solutions, for small k, can approximately be equal to the double wave; consequently, the problem reduces to the solution of equations (1.14) and (2.12) with the initial conditions given above. A cruder approximation may be obtained by considering instead of the system (1.14) and (2.12), Equations (1.20) and (1.21) with the corresponding initial data and $h = O(k)$.

This allows us to construct an approximate flow pattern in the neighborhood of a weak discontinuity in many problems, for example, when the weak discontinuity results from the motion of a curved piston of arbitrary form out of the volume of gas at rest according to some law of motion, and the flow is such that none of the perturbations in the flow region reach the stage of the weak discontinuity.

Note 3.1. The problems considered in the class of double waves for the system of equations (1.14) and (2.12) permit us to construct the flows behind weak discontinuities not only when the weak discontinuity becomes a sound wave as $t \to \infty$, but also for many flows whose existence is limited in time, and which are valid only until a shock wave forms. This can be done, for example, as in [17], using the method of reflecting the flows in question in the coordinate axes. There are four possible types of such flows and they are analogous to the one-dimensional case of centered simple waves, where 'forward-facing' and 'backward-facing' expansion and compression waves occur (cf [17]).

Note 3.2. In the use of double waves in the approximate construction of flows in the neighborhood of arbitrary weak discontinuities, it may be useful to point out the fact that the jumps in the derivatives of u_1, u_2, and c are at each instant inversely proportional to the radius of curvature of the line of the weak discontinuity. This fact easily follows from formulas (3.24), which give the line of weak discontinuity for the values t = const, $r = 0$, in a parametric form.

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